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NOTE

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Generation of exactly solvable quantum mechanical potentials

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Abstract : A method of generation of some exactly solvable potentials, using the properties of classical orthogonal polynomials, is presented. Some of these potentials are new. The same method can be used to generate different Sturmian potentials.

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Potentials governing quantum systems (QS) very often do not yield exact solutions of the Schrödinger equation. This warrants application of some suitable approximation schemes. The ease and economy in getting approximate analytical solutions largely depend on the similarity of the potential to some exactly solvable potentials. Besides, exact solutions of as yet undiscovered potentials may lead to new physics in the realm of non-relativistic quantum mechanics. With the advent of Supersymmetric (SUSY) quantum mechanics and its connection to solvable potential through Gendenshtein's 'Shape invariance' of SUSY partner potentials [1], it becomes relatively easy to find some exactly solvable potentials (ESP). Many authors have contributed new ESPs [2–4] that include QS's such as quasi-exactly solvable QS's [5–9], conditionally exactly solvable QS's [10,11], conditionally quasi-exactly solvable QS's [12] ans QS's generated through a transformation method [13,14].

In this Note, we put forward an alternative prescription to generate exactly solvable spherically symmetric potentials. The method is based on properties of classical orthogonal polynomials. A two-factor wave function

$$\psi(r) = F(g(r))Q_n(g(r)), \tag{1}$$

is considered, each factor of which is assumed to be a function of $g(r)$, where $g(r)$ is a smooth function of r. The radial Schrödinger equation is established in N_D dimensional space in which the co-efficient of $\psi(r)$ is essentially an undefined function say, $\Omega(r)$. Considering the post-factor of $\psi(r)$ to be a known classical orthogonal polynomial, it is possible to reduce the expression for $\Omega(r)$ to a form $[A_n - B(g)]$, where A_n is a constant independent of r, and $B(g)$ is a function of $g(r)$. Consistency arguments then fix the analytical forms of $F(g(r))$ of $\psi(r)$ as well as that of $g(r)$. This allows conversion of $[A_n - B(g)]$ to the standard form $[E_n - V(r)]$ from which the energy eigenvalues E_n and the solved potential $V(r)$ are read off. As both the factors of $\psi(r)$ are specified, the wave function becomes known for the potential $V(r)$. The method gives us some well-known QS's including some new Sturmian QS's well-equipped with normalized wavefunctions and are listed in Tables 1 and 2.

If $Q_n(r)$ represents a n-th degree polynomial in r defined through the generalized Rodrigues' formula [15], $Q_n(r) = \frac{1}{W}\frac{d^n}{dr^n}(W(r)S^n(r))$, with the help of the weight $W(r)$ and an auxiliary function $S(r)$, then the set $\{Q_n(r)\}$, (n = 0, 1, 2, 3, ...) will form an orthogonal set with weight $W(r)$ on the interval $[a, b]$ provided $Q_1(r)$, $W(r)$ and $S(r)$ satisfy : (i) $Q_1(r)$ is a first degree polynomial in r; (ii) $S(r)$ is a polynomial in r of degree ≤ 2, with real roots of the

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Table 1. Specification of S, $W$and the domain of different classical orthognal polynomials which satisfy eq. (3), and the derived functional form of $g(r)$.

Polynomials	$S(g)$	$g(r)$	$W(g)$	Domain	Generated potentials $V(r)$
Lauguerre	g	$\frac{r^2}{4}$	$4g^{\beta_0-1}e^{\beta_1 g}$; $\beta_0>0$; $\beta_1<0$.	$[0,\infty]$	$V_1(r)$
Hypergeometric	$g(1-g)$	$\frac{1+\sin r}{2}$	$g^{\beta_0-1}(1-g)^{-(\beta_0+\beta_1+1)}$; $\beta_0>0$, $\beta_0+\beta_1<0$.	$[0,1]$	$V_2(r)$
Hypergeometric	$g(1-g)$	$\frac{1-\cos r}{2}$	$g^{\beta_0-1}(1-g)^{-(\beta_0+\beta_1+1)}$	$[0,1]$	$V_3(r)$
Gegenbaur	g^2-1	$\cosh r$	$(-1)^{\lambda-\frac{1}{2}}(1-g^2)^{\lambda-\frac{1}{2}}$; $\beta_0=0$, $\beta_1=2\lambda+1$, $\lambda-\frac{1}{2}>0$ and even no.	$[-1,1]$	$V_4(r)$
Legendre	$1-g^2$	$\sin r$	$(1+g)^{\frac{\beta_0-\beta_1}{2}-1}$ $(1-g)^{-\frac{\beta_0+\beta_1}{2}-1}$; $\beta_0=0$; $\beta_0=-2$.	$[-1,1]$	$V_5(r)$
Jacobi	$1-g^2$	$\sin r$	$(1+g)^{\beta'}(1-g)^{\alpha'}$; $\alpha'=-\frac{\beta_0+\beta_1}{2}-1$, $\beta'=\frac{\beta_0-\beta_1}{2}-1$, $\beta_0-\beta_1>0$, $\beta_0+\beta_1<0$.	$[-1,1]$	$V_6(r)$
Hermite	1	r	$e^{\beta_0 g+\frac{\beta_1}{2}g^2}$; $\beta_0=0$, $\beta_1=-2$.	$[-\infty,\infty]$	$V_7(r)$

Table 2. List of different exactly solved potentials.

$V(r)$	Potentials	E_n	ψ_n
$V_1(r)$	$\frac{r^2}{16}+\frac{l(l+1)}{r^2}$	$\frac{1}{2}\left(2n_r+l+\frac{3}{2}\right)$; $\beta_0=l+3/2$;	$e^{-\frac{r^2}{8}}L^l_{n-l-1}\left(\frac{r^2}{4}\right)$.
$V_2(r)$	$A_1\tan^2 r+A_2\sec r\tan r$; $A_1=\beta_0(\beta_1+\beta_0)+\frac{(\beta_1+1)^2}{2}+\frac{1}{4}$; $A_2=\frac{(\beta_1+1)^2}{2}+\beta_0(\beta_1+2)-\frac{1}{2}$;	$n(n-\beta_1-1)-\{\beta_0(\beta_0+\beta_1)+\frac{(\beta_1+1)^2}{4}+\frac{1}{4}\}$;	$\frac{1}{r}\frac{\{1+\sin r\}^{\frac{\beta_0}{2}-\frac{1}{4}}}{\{1-\sin r\}^{\frac{\beta_0+\beta_1}{2}+\frac{1}{4}}}$ ${}_2F_1\left(-n,n-\beta_1-1,\beta_0;\frac{1+\sin r}{2}\right)$.
$V_3(r)$	$A_1\cot^2 r-A_2\csc r\cot r$,	-do-	$\frac{1}{r}\frac{\{1+\cos r\}^{\frac{\beta_0}{2}-\frac{1}{4}}}{\{1+\cos r\}^{\frac{\beta_0+\beta_1}{2}+\frac{1}{4}}}$ ${}_2F_1\left(-n,n-\beta_1-1,\beta_0;\frac{1-\cos r}{2}\right)$.
$V_4(r)$	$B_1\coth^2 r$, $B_1=\frac{(\beta_1-1)(\beta_1-3)}{4}$;	$-\{n(n+\beta_1-1)+\frac{\beta_1-1}{2}\}$;	$\frac{1}{r}\{\sinh r\}^{\frac{\beta_1-1}{2}}$ $C_n^{\frac{\beta_1-1}{2}}(\cosh r)$.
$V_5(r)$	$-\frac{1}{4}\tan^2 r$	$n(n+1)+\frac{1}{2}$;	$\frac{1}{r}(\cos r)^{\frac{1}{2}}P_n(\sin r)$.
$V_6(r)$	$C_1\tan^2 r+C_2\sec r\tan r$; $C_1=\frac{1}{2}\left(\alpha'^2+\beta'^2-\frac{1}{2}\right)$; $C_2=\frac{1}{2}\left(\alpha'^2-\beta'^2\right)$;	$n(n+\alpha'+\beta'+1)+\frac{\alpha'+\beta'+1}{2}-\frac{(\alpha'-\beta')^2}{4}$;	$\frac{1}{r}(1+\sin r)^{\frac{1}{2}\left(\beta'+\frac{1}{2}\right)}$ $(1-\sin r)^{\frac{1}{2}\left(\alpha'+\frac{1}{2}\right)}P_n^{(\alpha',\beta')}(\sin r)$.
$V_7(r)$	r^2	$2n+1$	$e^{-\frac{r^2}{2}}H_n(r)$

equation, $S(r) = 0$; (iii) $W(r)$ is positive, real and integrable in the domain $[a, b]$ and the combination $W(r)S(r)$ satisfies the boundary condition

$$W(a)S(a) = 0 = W(b)S(b). \tag{2}$$

Further, $Q_n(r)$ will be called a classical orthogonal polynomial and would satisfy the second-order differential equation :

$$\left(S(r)W(r)Q_n'(r)\right)' = -\lambda_n W(r)Q_n(R), \tag{3}$$

where prime represents differentiation with respect to its argument and the constant λ_n is given by,

$$\lambda_n = -n\left[K_1 Q_1'(r) + \frac{1}{2}(n-1)S''(r)\right], \tag{4}$$

K_1 being a constant utilized for the standardization of the polynomials[1]. Corresponding associated orthogonal polynomials defined over same domain $[a,b]$ are obtained through the relation (except for Hermite and Legendre) [16],

$$Q_{n,m}(r) = (-1)^m S(r)^{\frac{m}{2}} \frac{d^m}{dr^m} Q_n(r),$$

where m is at most n and satisfies,

$$\left(W(r)S(r)Q_{n,m}'(r)\right)' = -\eta_{n,m} W(r)Q_{n,m}(r),$$

The radial Schrödinger wave equation satisfied by $\psi(r)$ in N_D dimension is

$$\frac{\psi''(r)}{\psi(r)} + \frac{(N_D-1)}{r}\frac{\psi'(r)}{\psi(r)} = g'^2(r)\frac{F''(g)}{F(g)} + \left[g''(r) + \frac{(N_D-1)}{r}g'(r)\right]\frac{F'(g)}{F(g)} + g'^2(r)\frac{Q_n''(g)}{Q_n(g)} + g'^2\left(\frac{g''(r)}{g'^2(r)} + \frac{(N_D-1)}{rg'} + 2\frac{F'(g)}{F(g)}\right)\frac{Q_n'(g)}{Q_n(g)}. \tag{5}$$

Eq. (3) can be re-written as

$$S(r)\frac{Q_n''(r)}{Q_n(r)} + S(r)\frac{d}{dr}(\ln S(r)W(r))\frac{Q_n'(r)}{Q_n(r)} + \lambda_n = 0. \tag{6}$$

Assuming $Q_n(g)$, a classical orthogonal polynomial in g, and comparing Q_n-dependent part of eq. (5) with eq. (6), we can identify

$$S(g) = g'^2(r) \tag{7}$$

and
$$S(g)\frac{Q_n''(g)}{Q_n(g)} + S(g)\frac{d}{dg}\left(\ln\left|(S(g))^{\frac{1}{2}} r^{N_D-1}F^2(g)\right|\right) \times \frac{Q_n'(g)}{Q_n(g)} = -\lambda_n. \tag{8}$$

Hence, eq. (5) reduces to

$$\frac{\psi''(r)}{\psi(r)} + \frac{(N_D-1)}{r}\frac{\psi'(r)}{\psi(r)} = S(g)\frac{F''(g)}{F(g)} + S(g)\frac{d}{dg}\left(\ln\left((S(g))^{\frac{1}{2}} r^{N_D-1}\right)\right)\frac{F'(g)}{F(g)} = -\lambda_n. \tag{9}$$

Comparing (8) with (6), we get

$$S(g)W(g) = (S(g))^{\frac{1}{2}} r^{N_D-1} F^2(g). \tag{10}$$

Choosing the functional form of $S(g)$ which must be a polynomial in g of degree ≤ 2 with real roots, we can fix $g(r)$ from eq. (7). The most general form is $\alpha_0 + \alpha_1 g + \alpha_2 g^2$, where α_i's are arbitrary constants, satisfying the reality condition on the roots of $S(g)$, *i.e.* $\alpha_1^2 \geq 4\alpha_0\alpha_2$. Hence from eq. (7), we can write

$$\int(\alpha_0 + \alpha_1 g + \alpha_2 g^2)^{-\frac{1}{2}} dg = \pm r + c, \tag{11}$$

c being an integration constant. $Q_1(g)$ being a first degree polynomial in g, we can write

$$Q_1(g) = S(g)\frac{d}{dg}\ln(SW) = \beta_0 + \beta_1 g, \tag{12}$$

where β_i's are constants fixed by the particular classical polynomial considered. Using eqs. (10) and (12), we get

$$F(g) = \frac{1}{r}S^{-\frac{1}{4}}eF(g) = S^{-\frac{1}{4}} r^{-\frac{(N_D-1)}{2}} e^{\frac{1}{2}\int\frac{Q_1(g)}{S(g)}dg}. \tag{13}$$

Knowing $F(g)$ and $Q_n(g)$, we can have the wavefunction $\psi(r)$ and the R.H.S. of eq. (5) which is in fact $(V - E_n)$.

Choosing $S(g)$ for known classical orthogonal polynomials, different QS's can be generated. As an illustration of the method, let

$$S(g) = 1 - g^2, \tag{14}$$

which gives the functional form of $g(r)$ as

$$g(r) = \sin r \text{ or } \cos r, \tag{15}$$

where $\cos r$ is discarded as it leads to non-normalizable wavefunction. The choice of $S(g)$ is made with a view to specify the orthogonal polynomial $Q_n(g)$ either as Jacobi polynomial or Legendre polynomial by taking appropriate value of the set $\{\beta_0, \beta_1\}$. Considering $N_D = 3$ in equation (13) we get

$$F(g) = \frac{1}{r}S^{-\frac{1}{4}} e^{\frac{1}{2}\int\frac{Q_1(g)}{S(g)}dg}. \tag{16}$$

Eqs. (10) and (16) yield

$$S(g)W(g) = (1+g)^{\frac{\beta_0-\beta_1}{2}}(1-g)^{\frac{\beta_0+\beta_1}{2}}. \tag{17}$$

The domain of the orthogonal polynomial is found to be $[-1,1]$ from eq. (17), provided the conditions, (i) $\beta_0 - \beta_1 > 0$ and (ii) $\beta_0 + \beta_1 < 0$ are satisfied. The weight function $W(g)$ for generalized Jacobi polynomial can be written from eq. (17) as

$$W(g) = (1+g)^{\beta'}(1-g)^{\alpha'}, \tag{18}$$

where
$$\beta' = \frac{\beta_0 - \beta_1}{2} - 1 \,;\, \alpha' = -\frac{\beta_0 + \beta_1}{2} - 1. \tag{19}$$

[1] $K_n = (-1)^n$ for Hermite, $(-2)^n n!$ for Jocobi and Legendre, $n!$ for Laguerre *etc.*

The Schrödinger equation for $N_D = 3$ is

$$\frac{\psi''(r)}{\psi(r)} + \frac{2}{r}\frac{\psi'(r)}{\psi(r)} = S(g)\frac{F''(g)}{F(g)} + S(g)\frac{d}{dg} \times \left(\ln\left((S(g))^{\frac{1}{2}} r^2 \right) \right) \frac{F'(g)}{F(g)} - \lambda_n. \tag{20}$$

Eqs. (4), (15), (19) and (16) reduce eq. (20) to

$$\psi''(r) + \frac{2}{r}\psi'(r) + [E_n - V(r)]\psi(r) = 0, \tag{21}$$

which represents a QS with energy eigenvalues, potential and normalized wavefunction as

$$E_n = n(n+\alpha'+\beta'+1) + \frac{\alpha'+\beta'+1}{(\alpha'-\beta')^2} \tag{22}$$

$$V(r) = \frac{1}{2}\left[\left(\alpha'^2 + \beta'^2 - \frac{1}{2}\right)\tan^2 r + (\alpha'^2 - \beta'^2)\sec r \tan r\right] \tag{23}$$

and $$\psi_n(r) = \frac{N}{r}(1+\sin r)^{\frac{1}{2}\left(\beta'+\frac{1}{2}\right)}(1-\sin r)^{\frac{1}{2}\left(\alpha'+\frac{1}{2}\right)} \times P_n^{(\alpha',\beta')}(\sin r), \tag{24}$$

respectively, where N is

$$\left(\frac{n!(\alpha'+\beta'+2n+1)\Gamma(\alpha'+\beta'+n+1)}{2^{\alpha'+\beta'+1}\Gamma(\alpha'+n+1)\Gamma(\beta'+n+1)} \right)^{\frac{1}{2}}$$

$Q_n(g)$ can be made to represent Legendre polynomial for the choices $\beta_0 = 0$, $\beta_1 = -2$, which make $W(g) = 1$, and eq. (21) reduces to

$$\psi''(r) + \frac{2}{r}\psi'(r) + \left[n(n+1) + \frac{1}{2} + \frac{1}{4}\tan^2 r\right]\psi(r) = 0 \tag{25}$$

representing a totally different QS with energy eigenvalues, potential and normalized wavefunction as $E_n = n(n+1) + \frac{1}{2}$, $V(r) = -\frac{1}{4}\tan^2 r$ and $\psi_n(r) = r^{-1}\left[\left(n+\frac{1}{2}\right)\cos r\right]^{\frac{1}{2}} P_n(\sin r)$ respectively.

Similarly, other classical orthogonal polynomials can also be utilized to obtain different ESP's by appropriate choices of $S(g)$ as indicated in Table 1. The corresponding energy eigenvalues, potentials and normalized wavefunctions are given in Table 2. Of these, V_1, V_5 and V_7 are already reported in the literature.

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